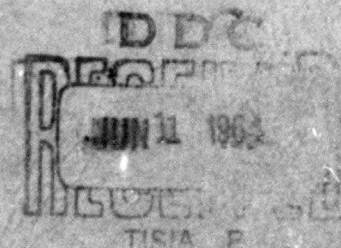13 p $0.50

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Bounds for the Rank of the Sum of Two Matrices

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BOUNDS FOR THE RANK OF THE SUM OF TWO MATRICES

by

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Summary

This paper gives upper and lower bounds on the rank of the sum of two matrices, and discusses their connection with the condition that rank be additive over a set of matrices.

1. Introduction

An important theorem in statistics, due to Cochran,[2], says that if $A_1, \ldots, A_m$ are symmetric $n \times n$ matrices for which

(I) $\quad A_1 + \cdots + A_m = I$

(II) $\quad \text{rank } (A_1) + \cdots + \text{rank } (A_m) = n,$

then $A_i A_j = 0$, $i \neq j$.

Various versions of this theorem have been considered, [1], [3], and [4]. Smiley [5] has recently offered another proof. The key point of the theorem seems to be that the rank of the sum is the sum of the ranks, and this note was undertaken with the intention of finding necessary and sufficient conditions that $r(A+B) = r(A) + r(B)$, for arbitrary A and B. Searching for such conditions led to looking for sharp upper and lower bounds on $r(A+B)$, and there did not seem to be much in the literature on this, other than the trivial $r(A+B) \leq r(A) + r(B)$. An improved upper bound and a lower bound, which seems to be new, were found; these are given in Section 4, Theorem 1. Conditions for $r(A+B) = r(A) + r(B)$, (Theorem 2), follow immediately from Theorem 1, so that Theorem 1 appears to be more important than the result we set out to prove, hence the title of this note.

As indicated above, we use $r(A)$ to designate the rank of the matrix A, and $d(\mathcal{A})$ for the dimension of the vector space $\mathcal{A}$. We also use I and 0 indiscriminately to represent the identity and zero matrix of whatever size necessary to fit the other matrices in a particular expression. In the usual way, for two vector spaces $\mathcal{R}_1$ and $\mathcal{R}_2$ we let $\mathcal{R}_1 + \mathcal{R}_2$ be the space of vectors $\rho_1 + \rho_2$ with $\rho_1 \in \mathcal{R}_1$ and $\rho_2 \in \mathcal{R}_2$. We also make extensive use of the projector of a vector space. The properties of a projector are easily established; we summarize them in the next section.

2. The projector of a space

Given a vector subspace $\mathcal{A}$ of the space of $1 \times n$ vectors, there is a unique symmetric, idempotent $n \times n$ matrix $\bar{A}$ satisfying

(I) $\alpha\bar{A} = \alpha$ if and only if $\alpha \in \mathcal{A}$.

(II) rank of $\bar{A}$ = dimension of $\mathcal{A}$.

This symmetric, idempotent matrix is called the __projector__ of $\mathcal{A}$.

If $\alpha_1, \alpha_2, \ldots, \alpha_t$ are an orthonormal basis of $\mathcal{A}$, then $\bar{A} = \alpha_1'\alpha_1 + \cdots + \alpha_t'\alpha_t$, or if A is a matrix whose rows are a basis of $\mathcal{A}$, then $\bar{A} = A'(AA')^{-1}A$. Any $1 \times n$ vector β may be represented uniquely as the sum of a vector in $\mathcal{A}$ and a vector in the orthogonal complement of $\mathcal{A}$, by writing $\beta = \beta\bar{A} + \beta(I - \bar{A})$. The projector of the orthogonal complement of $\mathcal{A}$ is $I - \bar{A}$, and $r(\bar{A}) + r(I - \bar{A}) = n$.

We will use an overscore to represent the projector--given a matrix B, the notation $\bar{B}$ will mean the projector of the row space of B; thus $\bar{B}$ is the unique matrix satisfying

(I) $\bar{B}' = \bar{B}$

(II) $\bar{B}\bar{B} = \bar{B}$

(III) $\bar{B} = TB$ for some T, i.e. the rows of $\bar{B}$ are in the row space of B.

(IV) $B\bar{B} = \bar{B}$

(V) $r(\bar{B}) = r(B)$.

The projector provides a convenient method for representing the rank of a composite matrix as a sum--for example, we write

$$r\binom{A}{B} = r\begin{pmatrix} A\bar{B}+A(I-\bar{B}) \\ B \end{pmatrix}$$

and since the rows of $A\bar{B}$ are in the row space of B, they may be removed by elementary row operations; thus

(1) $\qquad r\binom{A}{B} = r\begin{pmatrix} A(I-\bar{B}) \\ B \end{pmatrix} = r(B) + r[A(I-\bar{B})],$

since the rows of $A(I-\bar{B})$ are orthogonal to the rows of B.

For the column version of this formula, we write

$$r(R,S) = r(S) + r[(I-\overline{S'})R]$$

where $\overline{S'}$ is the projector of the column space of S, or the projector of the row space of S'.

A slightly more complicated application of this device gives this formula:

(2) $\qquad r\binom{RS}{TO} = r(T) + r(S) + r[(I-\overline{S'})R(I-T)]$

where $\overline{S'}$ is the projector of the column space of S and $\bar{T}$ the projector of the row space of T. To prove this, we write

$$r\binom{RS}{TO} = r\begin{pmatrix} R\bar{T}+R(I-\bar{T}) & S \\ T & 0 \end{pmatrix} = r\begin{pmatrix} R(I-\bar{T}) & S \\ T & 0 \end{pmatrix}$$

$$= r(T) + r(R(I-\bar{T}),S) = r(T) + r(S) + r[(I-\overline{S'})R(I-\bar{T})].$$

3. Bounds for the rank of a product

We need the results of this section to prove the main theorem. We want bounds on the rank of AB; to get them we write

$$r(B) = r\binom{B}{AB}$$

then use formula (1) to get

(3) $\qquad r(B) = r(AB) + r[B(I-\overline{AB})],$

where $\overline{AB}$ is the projector of the row space of AB. Now if $\overline{A}$ is the projector of the row space of A, then $\overline{A}B(I-\overline{AB}) = 0$, since for some T, $\overline{A} = TA$. Thus (3) may be written

$$r(B) = r(AB) + r[(I-\overline{A})B(I-\overline{AB})],$$

which gives these bounds on r(AB),

$$r(B) - r[(I-\overline{A})B] \leq r(AB) \leq r(B),$$

and the weaker result,

$$r(B) - r(I-\overline{A}) \leq r(AB) \leq r(B).$$

The latter is known as Sylvester's law of nullity, usually written as

$$(4) \qquad r(A) + r(B) - n \leq r(AB) \leq r(B)$$

where $r(I-\overline{A}) = n - r(\overline{A}) = n - r(A)$, assuming that A is p × n and B is n × q.

4. <u>Bounds for the rank of a sum</u>

This theorem gives upper and lower bounds for the rank of a sum.

<u>Theorem 1.</u> <u>Let A and B be two matrices of the same size, let their row spaces be</u> $\mathcal{R}_1$ <u>and</u> $\mathcal{R}_2$, <u>their column spaces</u> $\mathcal{C}_1$ <u>and</u> $\mathcal{C}_2$. <u>Then</u>

$$(5) \qquad r(A) + r(B) - d(\mathcal{R}_1 \cap \mathcal{R}_2) - d(\mathcal{C}_1 \cap \mathcal{C}_2) \leq r(A+B)$$

and

$$(6) \qquad r(A+B) \leq r(A) + r(B) - \max\{d(\mathcal{R}_1 \cap \mathcal{R}_2), d(\mathcal{C}_1 \cap \mathcal{C}_2)\}.$$

We first prove (6), which is quite easy, then (5), which is not so easy. We have

$$r(A+B) \leq d(R_1 + R_2) = d(R_1) + d(R_2) - d(R_1 \cap R_2).$$

Since $d(R_1) = r(A)$ and $d(R_2) = r(B)$, we have

$$r(A+B) \leq r(A) + r(B) - d(R_1 \cap R_2),$$

and a similar argument on the column spaces yields (6).

Using the fact that $r({}^A_B) = d(R_1 + R_2) = r(A) + r(B) - d(R_1 \cap R_2)$ and $r(A,B) = d(C_1 + C_2) = r(A) + r(B) - d(C_1 \cap C_2)$, we may write (5) in the form

$$(7) \qquad r({}^A_B) + r(A,B) \leq r(A+B) + r(A) + r(B).$$

Now none of the five ranks in (7) is changed if we replace A and B by PAQ and PBQ with P and Q nonsingular. Thus we may assume A and B have any form obtained by performing identical elementary row and column operations on each of them. We may, for example, assume that A and B have this form

$$A = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} R & S & 0 \\ T & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}.$$

In that case, we have $r({}^A_B) = r(I_1) + r(S) + r(I_2)$, and $r(A,B) = r(I_1) + r(T) + r(I_2)$, so that we will prove (7) if we can prove that

$$(8) \qquad r(I) + r(S) + r(T) \leq r({}^R_T {}^S_0) + r({}^{I+R}_T {}^S_0),$$

dropping the subscript on I, assumed to be $n \times n$. Using formula (2), we write

$$r\binom{R\ S}{T\ 0} + r\binom{I+R\ S}{T\ 0} = 2r(S) + 2r(T) + r[(I-\overline{S^{\intercal}})(I+R)(I-\overline{T})] + r[(I-\overline{S^{\intercal}})R(I-\overline{T})]$$

and this leads to the inequality

$$(9)\quad r\binom{R\ S}{T\ 0} + r\binom{I+R\ S}{T\ 0} \geq 2r(S) + 2r(T) + r[(I-\overline{S^{\intercal}})(I-\overline{T})],$$

since $r[(I-\overline{S^{\intercal}})R(I-\overline{T})] = r[(I-\overline{S^{\intercal}})(-R)(I-\overline{T})]$. Using (4) with $n = r(I)$, we have

$$r[(I-\overline{S^{\intercal}})(I-\overline{T})] \geq n - r(S) + n - r(T) - n$$

and putting this in (9) we have

$$r\binom{R\ S}{T\ 0} + r\binom{I+R\ S}{T\ 0} \geq r(S) + r(T) + n.$$

This establishes formula (8), and hence (7), which is equivalent to (5); the proof of Theorem 1 is complete.

5. The rank of the sum and the sum of the ranks.

Theorem 1 enables us to characterize the condition that rank be additive for a pair of matrices:

Theorem 2. Let A and B be two matrices of the same size, with row spaces R_1, R_2 and column spaces C_1, C_2. Then

$$r(A+B) = r(A) + r(B)$$

if, and only if

$$\dim(R_1 \cap R_2) = \dim(C_1 \cap C_2) = 0.$$

The proof follows immediately from (5) and (6).

The fact that rank is additive for a set of matrices is a strong condition. We will develop some consequences of this condition in a

few conclusions leading up to Theorem 3, which is a sort of generalized Cochran's Theorem. First, we point out that if rank is additive on a set of matrices, it is additive on subsets:

Lemma 1. If rank is additive for a set of matrices;

$$r(A_1 + A_2 + \cdots + A_m) = r(A_1) + r(A_2) + \cdots + r(A_m),$$

then rank is additive for sets of matrices formed by adding distinct A's from that set -- for example, if $r(A_1 + \cdots + A_{12}) = r(A_1) + \cdots + r(A_{12})$, then

$$r(A_1+A_5)+r(A_3+A_8+A_{10})+r(A_6+A_9) = r(A_1+A_5+A_3+A_8+A_{10}+A_6+A_9).$$

To give the gist of the proof, consider the example of the theorem. Let $A = A_1+A_5$, $B = A_3+A_8+A_{10}$, $C = A_6+A_9$, and let D be the sum of the matrices in $A_1,\ldots,A_{12}$ not included in A, B, or C, that is, $D = A_2+A_4+A_7+A_{11}+A_{12}$. Then

$$r(A_1)+\cdots+r(A_{12}) = r(A+B+C+D)\leq r(A+B+C)+r(D)\leq r(A)+r(B)+r(C)+r(D)\leq r(A_1)+\cdots+r(A_{12}).$$

Thus all inequalities are equalities, and $r(A+B+C) = r(A) + r(B) + r(C)$. The converse of this lemma is not true -- rank can be pairwise additive but yet not finitely additive, for example, for these three positive semi-definite matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix},$$

rank is pairwise additive but not additive over all three.

Lemma 2. If $r(A+B) = r(A) + r(B)$ and if A and B each commute with their sum, $S = A+B$, then $AB = BA = 0$.

Proof: Since A commutes with S, it commutes with $S - A = B$. Since the row space of $AB = BA$ is in both the row space of A and the row space of B, Theorem 2 shows that $AB = BA = 0$.

We are now able to give this generalized version of Cochran's Theorem mentioned in the Introduction. We find that symmetry has no essential role in the theorem, and Condition II, that $\Sigma A_i = I$, can be replaced by the condition that each A_j commutes with ΣA_i.

__Theorem 3.__ __Let__ $A_1, A_2, \ldots, A_m$ __be square matrices for which rank is additive,__

$$r(A_1 + \cdots + A_m) = r(A_1) + \cdots + r(A_m),$$

__and let__ S __be their sum:__ $S = A_1 + \cdots + A_m$. __Then__

$$(10) \qquad A_i A_j = 0 \qquad \text{for } i \neq j$$

__if and only if each of the__ A's __commutes with the sum:__

$$(11) \qquad A_i S = S A_i, \qquad i = 1, 2, \ldots, m.$$

Proof: Condition (10) certainly implies (11). To prove that (11) implies (10) we prove that $A_1 A_2 = A_2 A_1 = 0$. Let $A = A_1$, $B = A_2$, and $C = A_3 + \cdots + A_m$. Using Lemmas 1 and 2, we know that $A(B+C) = (B+C)A = 0$, and hence $A^2 = AS = SA$. Going to a similarity transformation if necessary, we may assume that A has the form $\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$, with G non-singular. Then we write $A + B + C = S$ in the form

$$\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} .$$

Since $A^2 = AS = SA$ and G is non-singular, it follows that S must have the form $\begin{pmatrix} G & 0 \\ 0 & S_4 \end{pmatrix}$. Thus

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S_4 \end{pmatrix}.$$

Now according to Lemma 1, $r(B+C) = r(B) + r(C)$, and it follows that the rows of $(B_1 \ B_2)$ and of $(C_1 \ C_2)$ must all be zero, or else Theorem 2 would be violated. By the same argument, the columns of $\begin{pmatrix} B_1 \\ B_3 \end{pmatrix}$ and $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ must be zero. Thus $B = \begin{pmatrix} 0 & 0 \\ 0 & B_4 \end{pmatrix}$, and $AB = BA = 0$.

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